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# Chapter = Complex Numbers

$i = \sqrt{-1}$ $\quad i^2 = -1$

$i^3 = -i$ $\quad i^4 = 1$

Complex numbers $= a + ib$

$a, b \in R$

## Exercise - 5.1

Express . . . . . . . . . . . . $a+ib$

1\. $(5i)\left(-\frac{3i}{5}\right)$

$\cancel{5} i \times \frac{(-3i)}{\cancel{5}} \Rightarrow i \times -3i$

$-3i^2 \Rightarrow -3 \times -1 \Rightarrow 3$

$3 + i \cdot 0$ $[a+ib]$

2\. $i^9 + i^{19}$

$i^9 (1 + \cancel{10}\, i^{10})$

$(i^3)^3 [1 + (i^2)^5]$

$(i)^3 \; [1-1] = 0$

$0 + i0$ $[a+ib]$

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3. $i^{-39}$

$\frac{1}{i^{39}} \times \frac{i}{i} \Rightarrow \frac{i}{i^{40}} \Rightarrow \frac{i}{(i^4)^{10}} = \frac{i}{1}$

$0 + i \cdot 1$ [a + ib]

5. $(1-i) - (-1 + i6)$

$1 - i + 1 - i6$
$2 - i7$ [a+ib]

7. $\left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right)\right] - \left(\frac{-4}{3} + i\right)$

$\left[\frac{1}{3} + \frac{7}{3}i + 4 + \frac{1}{3}i\right] + \frac{4}{3} - i$

$\frac{13}{3} + \frac{8i}{3} + \frac{4}{3} - i$

$\frac{17}{3} + \frac{5i}{3}$ [a+ ib] $\Rightarrow \frac{17}{3} + i\frac{5}{3}$ [a+ib]

9. $\left(\frac{1}{3} + 3i\right)^3$

$\left(\frac{1}{3}\right)^3 + (3i)^3 + 3 \times \frac{1}{3} \times 3i\left(\frac{1}{3} + 3i\right)$

$\frac{1}{27} - 27i + 1 + 9i^2$

$$\frac{1}{27} - 26i - 9 \Rightarrow \frac{1}{27} - 9 - 26i$$

$$\frac{1-243}{27} - 26i \Rightarrow -\frac{242}{27} - i26$$

$$-\frac{242}{27} - i26 \left[a+ib\right]$$

11. $4-3i$

Multiplicative inverse $= \frac{1}{4-3i}$

$$\frac{1}{4-3i} \times \frac{4+3i}{4+3i} \Rightarrow \frac{4+3i}{(4)^2-(3i)^2}$$

$$\Rightarrow \frac{4+3i}{16-9(i^2)} \Rightarrow \frac{4+3i}{16-9(-1)}$$

$$= \frac{4+3i}{16+9}$$

$$= \frac{4+3i}{25}$$

$$= \boxed{\frac{4}{25} + i\frac{3}{25}}$$

14. Express in the form a+ib

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$$

$$\frac{(3)^2-(i\sqrt{5})^2}{\cancel{\sqrt{3}}+\sqrt{2}i-\cancel{\sqrt{3}}+i\sqrt{2}}$$

$$\Rightarrow \frac{9-(-1)5}{i2\sqrt{2}} \times \frac{i}{i} \Rightarrow \frac{9+5}{i2\sqrt{2}} \times \frac{i}{i}$$

$$\Rightarrow \frac{\cancel{14}\,7}{i\cancel{2}\sqrt{2}} = \frac{7i}{i^2\sqrt{2}} \Rightarrow -\frac{7i}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$\frac{-7\sqrt{2}i}{2} \Rightarrow \boxed{0-i\frac{7\sqrt{2}}{2}} \quad a+ib$$

4. $3(7+i7)+i(7+i7)$

$21+i21+i7+i^27$

$21+i28-7$

$14+i28$ [a+ib]

6. $\left[\frac{1}{5}+i\frac{2}{5}\right]-\left[4+i\frac{5}{2}\right]$

$\frac{1}{5}+i\frac{2}{5}-4-i\frac{5}{2}$

$\frac{1}{5}-4+i\left[\frac{2}{5}-\frac{5}{2}\right]$

$$\frac{1-20}{5} + i\left[\frac{4-25}{10}\right]$$

$$\frac{-19}{5} + i\frac{(-21)}{10} \Rightarrow \boxed{\frac{-19}{5} - i\frac{21}{10}}$$

8. $(1-i)^4$

$$[(1-i)^2]^2$$
$$[1+i^2-2i]^2$$
$$[\cancel{1}-\cancel{1}-2i]^2$$
$$[-2i]^2$$
$$4i^2$$
$$4(-1) \Rightarrow -4$$
$$\boxed{-4+i0 \; [a+ib]}$$

10. $\left[-2-\frac{1}{3}i\right]^3$

$$(-2)^3 - \left(\frac{1}{3}i\right)^3 - 3 \times (-2) \times \left(\frac{1}{3}i\right)\left[-2-\frac{1}{3}i\right]$$

$$-8 - \frac{1}{27}i^3 + 2i\left[-2-\frac{1}{3}i\right]$$

$$-8 + \frac{1}{27}i + 2i\left[-2-\frac{1}{3}i\right]$$

$$-8 + \frac{1}{27}i - 4i - \frac{2}{3}i^2$$

$$-8+\frac{2}{3}+\frac{1}{27}i-4i$$

$$\frac{-24+2}{3}+\frac{i-108i}{27}$$

$$\frac{-22}{3}-\frac{107i}{27}$$

$$\Rightarrow \boxed{\frac{-22}{3}-i\frac{107}{27}}$$

12. $\sqrt{5}+3i$

Multiplication inverse $=\frac{1}{\sqrt{5}+3i}$

$$\frac{1}{\sqrt{5}+3i}\times\frac{\sqrt{5}-3i}{\sqrt{5}-3i}\Rightarrow\frac{\sqrt{5}-3i}{(\sqrt{5})^2-(3i)^2}$$

$$\Rightarrow\frac{\sqrt{5}-3i}{5-9i^2}\Rightarrow\frac{\sqrt{5}-3i}{5+9}\Rightarrow\frac{\sqrt{5}-3i}{14}$$

$$\Rightarrow\boxed{\frac{\sqrt{5}}{14}-i\frac{3}{14}}$$ Ans

13. $-i$

Multiplication inverse of $-i=\frac{1}{-i}$

$$\Rightarrow\frac{1}{-i}\times\frac{i}{i}\Rightarrow\frac{i}{-i^2}\Rightarrow\frac{i}{-(-1)}\Rightarrow\frac{i}{+1}$$

$$\Rightarrow\boxed{i}$$ Ans

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# Miscellaneous Exercise

1. Evaluate $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$

$\Rightarrow \left[i^{16} \times i^2 + \left(\frac{1}{i} \times \frac{i}{i}\right)^{25}\right]^3$

$\Rightarrow \left[1 \times (-1) + (-i)^{25}\right]^3$

$\Rightarrow \left[-1 - i^{25}\right]^3$

$\Rightarrow \left[-1 - i^{24} \times i\right]^3 \Rightarrow \left[-1 - (i^4)^6\, i\right]^3$

$\Rightarrow \left[-1 - i\right]^3$

$\Rightarrow (-1)^3 + (-i)^3 + 3 \times (-1)(-i)(-1 - i)$

$\Rightarrow -1 + -i^3 + 3i(-1 - i)$

$-1 + i - 3i - 3i^2 \Rightarrow -3i^2 - i - 2i$

$-3(-1) - 2i \Rightarrow \boxed{3 - 2i}$

4. If . . . . . . . . . . . . . . $\frac{a^2+b^2}{c^2+d^2}$

$x - iy = \sqrt{\frac{a - ib}{a - id}}$ - eq (1)

$x + iy = \sqrt{\frac{a + ib}{c + id}}$ - eq (2)

eq (1) × eq (2)

$(x - iy)(x + iy) = \sqrt{\frac{(a - ib)(a + ib)}{(c - id)(c + id)}}$

$x^2 - i^2 y^2 = \sqrt{\frac{a^2 - i^2 b^2}{c^2 - i^2 d^2}}$

$$x^2+y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}} \Rightarrow (x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$$

10. If $z_1 = 2-i$, $z_2 = 1+i$, find $\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right|$

$$\left|\frac{2-\cancel{i}+1+\cancel{i}+1}{2-i-\cancel{1}-i+\cancel{1}}\right| \Rightarrow \left|\frac{\cancel{4}^{2}}{\cancel{2}(1-i)} \times \frac{(1+i)}{(1+i)}\right|$$

$$\Rightarrow \left|\frac{2(1+i)}{(1)^2-(i)^2}\right| \Rightarrow \left|\frac{2(1+i)}{1-i^2}\right| \Rightarrow \left|\frac{2(1+i)}{1-(-1)}\right|$$

$$\Rightarrow \left|\frac{2(1+i)}{1+1}\right| \Rightarrow \left|\frac{\cancel{2}(1+i)}{\cancel{2}}\right| \Rightarrow |(1+i)|$$

$$\Rightarrow \sqrt{(1)^2+(1)^2} \Rightarrow \sqrt{1+1} \Rightarrow \sqrt{2}$$

$$\boxed{\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right| = \sqrt{2}}$$

12. Let $z_1 = 2-i$, $z_2 = -2+i$ find

(i) Re $\left(\frac{z_1 z_2}{\bar{z}_1}\right)$

$\bar{z}_1 = 2+i$

$$\frac{z_1+z_2}{\bar{z}_1} \Rightarrow \frac{(2-i)(-2+i)}{(2+i)}$$

$$\frac{-(2-i)\ (+2-i)\ (2-i)}{(2+i)\ (2-i)}$$

$$\Rightarrow \frac{-(2-i)^3}{(2)^2-i^2} \Rightarrow \frac{-(8-i^3-6i(2-i))}{4-i^2}$$

$$\Rightarrow \frac{-(8-(-i)-12i+6i^2)}{4-(-i)}$$

$$\Rightarrow \frac{-(8+i\ 12i-6)}{4+1}$$

$$\Rightarrow \frac{-(8-11i-6)}{5} \Rightarrow \frac{-(2-11i)}{5}$$

$$\Rightarrow \boxed{\frac{-2}{5}+\frac{11i}{5}}$$

$$\boxed{Re\ \frac{z_1 z_2}{\overline{z_1}} = \frac{-2}{5}}$$

16. If $(x+iy)^3 = u+iv$, then show that $\frac{u}{x}+\frac{v}{y} = 4(x^2-y^2)$

$(x+iy)^3 = u+iv$
$x^3+i^3y^3+3x(iy)(x+iy) = u+iv$
$x^3-iy^3+i3x^2y+i^2 3xy^2 = u+iv$
$x^3-3xy^2+i3x^2y-iy^3 = u+iv$

$(x^3-3xy^2)+i(3x^2y-y^3) = u+iv$
On comparing
$u = x^3-3xy^2$
$u = x(x^2-3y^2)$

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$$\frac{u}{x} = x^2 - 3y^2 \quad - (1)$$

$$v = 3x^2y - y^3$$
$$v = (3x^2 - y^2)\, y$$
$$\frac{v}{y} = 3x^2 - y^2 \quad - (2)$$

eq (1) + eq (2)

$$\frac{u}{x} + \frac{v}{y} = x^2 - 3y^2 + 3x^2 - y^2$$
$$\frac{u}{x} + \frac{v}{y} = 4x^2 - 4y^2$$

$$\boxed{\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)}$$

17. If - - - - - - - - - - - - $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$

$\alpha = a + ib$ $\qquad\qquad$ $\beta = c + id$

$|\beta| = 1 \Rightarrow |c + id| = 1 \Rightarrow \sqrt{c^2 + d^2} = 1$

$c^2 + d^2 = 1 \quad - eq (1)$

$$\Rightarrow \left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| \Rightarrow \left|\frac{c + id - a - ib}{1 - (a - ib)(c + id)}\right|$$

$$\Rightarrow \left|\frac{(c - a) + i(d - b)}{1 - ac - iad + ibc + i^2 bd}\right|$$

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$$\Rightarrow \left| \frac{(c-a)+i(d-b)}{1-ac-bd+i(bc-ad)} \right|$$

$$\Rightarrow \frac{\sqrt{(c-a)^2+(d-b)^2}}{\sqrt{(1-ac-bd)^2+(bc-ad)^2}}$$

$$\Rightarrow \frac{\sqrt{c^2+a^2-2ac+d^2+b^2-2bd}}{\sqrt{1+a^2c^2+b^2d^2-2ac+\cancel{2abcd}-2bd+b^2c^2+a^2d^2-\cancel{2abcd}}}$$

$$\Rightarrow \frac{\sqrt{c^2+d^2+a^2+b^2-2ac-2bd}}{\sqrt{1+a^2c^2+a^2d^2+b^2c^2+b^2d^2-2ac-2bd}}$$

$$\Rightarrow \frac{\sqrt{1+a^2+b^2-2ac-2bd}}{\sqrt{1+a^2(c^2+d^2)+b^2(c^2+d^2)-2ac-2bd}}$$

$$\Rightarrow \frac{\sqrt{1+a^2+b^2-2ac-2bd}}{\sqrt{1+a^2(1)+b^2(1)-2ac-2bd}}$$

$$\Rightarrow \sqrt{\frac{\cancel{1+a^2+b^2-2ac-2bd}}{\cancel{1+a^2+b^2-2ac-2bd}}}$$

$$\Rightarrow 1$$

$$\Rightarrow \boxed{\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = 1}$$

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19. If $(a+ib)$ - - - - - - - - - - - - $A^2+B^2$

$$(a+ib)(c+id)(e+if)(g+ih) = A+iB \text{ - eq (1)}$$
$$(a-ib)(c-id)(e-if)(g-ih) = A-iB \text{ - eq (2)}$$

eq (1) & × (2)

$$(a^2-i^2b^2)(c^2-i^2d^2)(e^2-i^2f^2)(g^2-i^2h^2) = A^2-i^2B^2$$

$$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$$

Hence proved

18. Find - - - - - - - - - - - - - - $2^x$

$$|1-i|^x = 2^x$$
$$(\sqrt{1^2+1^2})^x = 2x$$
$$(\sqrt{2})^x = 2^x$$
$$2^{x/2} = 2^x$$
$$1 = \frac{2^x}{2^{x/2}} \Rightarrow 1 = 2^{x-\frac{x}{2}}$$
$$1 = 2^{x/2} \Rightarrow 2^0 \Rightarrow 2^{x/2}$$
$$\frac{x}{2} = 0 \qquad x = 0$$

2. For - - - - - - - - - - - - - - $Im z^2$

$$z_1 = a+ib$$
$$z_2 = c+id$$
$$z_1z_2 = (a+ib)(c+id) \Rightarrow ac + iad + ibc + i^2bd$$
$$\Rightarrow (ac-bd) + i(ab+bc)$$
$$Re(z_1 z_2) = ac - bd \text{ - (1)}$$

$$Re\, z_1\, Re\, z_2 - Im\, z_1\, Im\, z_2$$
$$= ac - bd \quad \text{— (2)}$$

comparing eq (1) & (2)

$$Re(z_1 z_2) = Re\, z_1\, Re\, z_2 - Im\, z_1\, Im\, z_2$$

Hence proved

3. Reduce - - - - - - - - - - - form

$$\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$$

$$\Rightarrow \left[\frac{1+i-2(1-4i)}{(1-4i)(1+i)}\right]\left[\frac{3-4i}{5+i} \times \frac{5-i}{5-i}\right]$$

$$\Rightarrow \left[\frac{1+i-2+8i}{1+i-4i-4i^2}\right]\left[\frac{15-3i-20i+4i^2}{(5)^2-(i)^2}\right]$$

$$\Rightarrow \left[\frac{-1+9i}{1+4-3i}\right]\left[\frac{15-23i-4}{25-(-1)}\right]$$

$$\Rightarrow \left[\frac{-1+9i}{5-3i}\right]\left[\frac{11-23i}{25+1}\right]$$

$$\Rightarrow \left[\frac{-1+9i}{5-3i} \times \frac{5+3i}{5+3i}\right]\left[\frac{11-23i}{26}\right]$$

$$\Rightarrow \left[\frac{-5-3i+45i+27i^2}{25-9i^2}\right]\left[\frac{11-23i}{26}\right]$$

$$\Rightarrow \left[\frac{-5-27+42i}{25+9}\right]\left[\frac{11-23i}{26}\right]$$

$$\Rightarrow \left[\frac{-32+42i}{34}\right]\left[\frac{11-23i}{26}\right]$$

$$\Rightarrow \cancel{2}\left[\frac{-16+21i}{\cancel{34}\;17}\right]\left[\frac{11-23i}{26}\right]$$

$$\Rightarrow \left[\frac{-176+368i+231i-483i^2}{442}\right]$$

$$\Rightarrow \left[\frac{-176+483+599i}{442}\right] \Rightarrow \frac{307+401i}{442}$$

4\. If $x+iy$ - - - - - - - - - - $\frac{a^2+b^2}{c^2+d^2}$

$$x+iy = \sqrt{\frac{a-ib}{c-id}}$$

<u>S.B.S.</u>

$$(x+iy)^2 = \frac{a-ib}{c-id}$$

$$\Rightarrow x^2+i^2y^2+2xyi = \frac{a-ib}{c-id}$$

$$\Rightarrow x^2-y^2+2xyi = \frac{a-ib}{c-id}$$

$$\Rightarrow |x^2-y^2+2xyi| = \left|\frac{a-ib}{c-id}\right|$$

$$\Rightarrow \sqrt{(x^2-y^2)+(2xy)^2} = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

S.B.S. $\Rightarrow (x^2-y^2)+(2xy)^2 = \frac{a^2+b^2}{c^2+d^2}$

$$\Rightarrow x^4+y^4-2x^2y^2+4x^2y^2 = \frac{a^2+b^2}{c^2+d^2}$$

$$\Rightarrow x^4+y^4+2x^2y^2 = \frac{a^2+b^2}{c^2+d^2}$$

$$(x^2+y^2) = \frac{a^2+b^2}{c^2+d^2}$$ Hence proved

11. If - - - - - - - - - - - - $a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$

$$a+ib = \frac{(x+i)^2}{2x^2+1}$$

$$\Rightarrow a+ib = \frac{x^2+i^2+2xi}{2x^2+1}$$

$$\Rightarrow a+ib = \frac{x^2-1+2xi}{2x^2+1}$$

$$\Rightarrow |a+ib| = \left|\frac{x^2-1+2xi}{2x^2+1}\right|$$

$$\Rightarrow \sqrt{a^2+b^2} = \frac{\sqrt{(x^2-1)^2+(2x)^2}}{2x^2+1}$$

$$\Rightarrow \text{S.B.S.},\ a^2+b^2 = \frac{x^4+1-2x^2+4x^2}{(2x^2+1)^2}$$

$$\Rightarrow a^2+b^2 = \frac{x^4+1+2x^2}{(2x^2+1)^2}$$

$$\Rightarrow a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$$ Hence proved

12(ii) $Im\left(\frac{1}{z_1 \bar{z_1}}\right)$

$z_1 = 2-i \quad \bar{z_1} = 2+i$

$z_1 \bar{z_1} \Rightarrow (2-i)(2+i)$

$\Rightarrow 2^2 - i^2 \Rightarrow 4-(-1)$

$\Rightarrow 4+1=5$

$\Rightarrow 5+0i$

$$\frac{1}{z_1 \bar{z_1}} = \frac{1}{5} = \frac{1}{5}+0i$$

$\boxed{Im\left(\frac{1}{z_1 \bar{z_1}}\right) = 0}$ Ans

13. find - - - - - - - - - - - - $\frac{1+2i}{1-3i}$

$$\frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} \Rightarrow \frac{1+3i+2i+6i^2}{(1)^2-(3i)^2}$$

$$\Rightarrow \frac{1-6+5i}{1-9i^2} \Rightarrow \frac{-5+5i}{1+9} \Rightarrow \frac{\cancel{5}(-1+i)}{\cancel{10}\,2}$$

$$\Rightarrow \left|\frac{-1}{2}+\frac{i}{2}\right| \Rightarrow \sqrt{\left(\frac{-1}{2}\right)^2+\left(\frac{1}{2}\right)^2}$$

$$\Rightarrow \sqrt{\frac{1}{4}+\frac{1}{4}} \Rightarrow \sqrt{\frac{\cancel{2}\,1}{\cancel{4}\,2}} \Rightarrow \sqrt{\frac{1}{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \Rightarrow \frac{\sqrt{2}}{2}$$ Ans

14. Find - - - - - - - - - - - 6-24i.

$$\overline{-6-24i} = (x+iy)(3+5i)$$

$$-6+24i = 3x+5xi-3yi-5yi^2$$

$$-6+24i = 3x+5y+(5x-3y)i$$

Comparing real part

$$3x+5y=-6 \quad -(1)$$

comparing Imaginary part

$$5x-3y=24 \quad -(2)$$

eq (1) × 3

$$9x+15y=-18$$

eq (2) × 5

$$25x-15y=120$$

$$25x-15y=120$$
$$9x+15y=-18$$

$$34x=102$$
$$x=3$$

$$3x+5y=-6$$
$$3(3)-5y=-6 \Rightarrow 9+6=-5y$$
$$15=-5y \qquad y=-3$$
$$x=3, \quad y=-3$$

15. Find . . . . . . . . . . . $\frac{1-i}{1+i}$

$$\frac{1+i}{1-i} - \frac{1-i}{1+i} \Rightarrow \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)}$$

$$\Rightarrow \frac{\cancel{1}+i^2+2i-\cancel{1}-i^2+2i}{(1)^2-(i)^2} = \frac{4i}{1-i^2}$$

$$\Rightarrow \frac{4i}{1+1} \Rightarrow \frac{\cancel{4}^{2}i}{\cancel{2}} \Rightarrow 2i$$

$$\Rightarrow |2i| \Rightarrow \sqrt{(2)^2} \Rightarrow \pm 2$$

20. If $\left(\frac{1+i}{1-i}\right)^m$ . . . . . of m

$$\left(\frac{1+i}{1-i}\right)^m = 1 \Rightarrow \left[\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right]^m = 1$$

$$\Rightarrow \left[\frac{(1+i)^2}{(1)^2-(i)^2}\right]^m = 1 \Rightarrow \left[\frac{1+i+2i}{1-i^2}\right]^m = 1$$

$$\Rightarrow \left[\frac{\cancel{1}-\cancel{1}+2i}{1+1}\right]^m = 1 \Rightarrow \left[\frac{\cancel{2}i}{\cancel{2}}\right]^m = 1$$

$$\Rightarrow [i]^m = 1 \Rightarrow (i)^m = (i)^4$$

$$\boxed{m = 4}$$